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ON THE STABILITY OF A NONAUTONOMOUS HAMILTONIAN SYSTEM UNDER A PARAMETRIC RESONANCE OF ESSENTIAL TYPE*

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The problem of the stability of the equilibrium position of an nonautonomous Hamiltonian system with periodic coefficients, in which two multipliers of the linearized system are equal, is analyzed in a nonlinear setting. The stability in the finite approximation, and formal Liapunov stability or instability are proved, depending on the Hamiltonian's coefficients.

1. We consider a nonautonomous Hamiltonian system with two degrees of freedom

$$\frac{dq_k}{dt} = \frac{\partial H}{\partial p_k}, \quad \frac{dp_k}{dt} = -\frac{\partial H}{\partial q_k} \quad (k = 1, 2) \tag{1.1}$$

whose Hamiltonian $H = H(q_k, p_k, t)$ is analytic in q_k, p_k in a neighborhood of the trivial equilibrium position

$$H = H_2 + \ldots + H_m + \ldots \tag{1.2}$$

where the H_m are mth-degree homogeneous polynomials in q_k, p_k with 2π-periodic and t-continuous coefficients $h_{\nu_1\nu_2\mu_1\mu_2}(t)$. The stability problem for such a system has been almost completely solved by now /1,2/. The case which in applied problems corresponds to the so-called parametric resonance of essential type /3/ remains unsolved and, as a rule, corresponds to the boundary of the stability region of the linearized system. The study of this case is necessitated by the desire to have a complete solution to the stability problem in concrete applied problems of mechanics. An example is the stability problem for the triangular libration points of the flat elliptic restricted three-body problem under bounded values of eccentricity and mass ratio. Problems of investigating the arbitrary periodic motions in autonomous Hamiltonian systems with the use of isopower reduction lead to systems of the type being analyzed.

At first we study the normalization of the linearized system with Hamiltonian H_2. In the case being examined, without loss of generality we can assume that a linear canonic transformation separating the variables has already been made in the system and that the function H_2 has been reduced to the form

$$H_2 = h_2(q_1, p_1) + {}^1/_2\delta_2\lambda_2(q_2^2 + p_2^2) \quad (\delta_2 = \pm 1, \ \lambda_2 > 0) \tag{1.3}$$

Therefore, for the present we take the original system to have one degreee of freedom and we consider it in detail.

Let $\mathbf{X}(t)$ be the matrix of fundamental solutions of a linear system with Hamiltonian $h_2(q_1, p_1)$, normed by the initial condition $\mathbf{X}(0) = \mathbf{E}$ ($\mathbf{E}$ is the unit matrix). Then under parametric resonance of basic type both eigenvalues of matrix $\mathbf{X}(2\pi)$ (i.e., the multipliers ρ, viz., the roots of the characteristic equation $\det\|\mathbf{X}(2\pi) - \rho\mathbf{E}\| = 0$) are real, equal to each other, and equal to ± 1. This signifies that the pure imaginary parts of the characteristic exponents $\pm i\lambda_1$ $(\rho = \exp(\pm 2\pi i\lambda_1))$ are integers of half-integers. In addition, since the matrix $\mathbf{X}(2\pi)$ has multiple eigenvalues, its normal form (and, consequently, the normal form of the Hamiltonian) depends upon the multiplicities of the elementary divisors of the characteristic matrix $\mathbf{X}(2\pi) - \rho\mathbf{E}$. Thus, we have to distinguish four cases: 1) $2\lambda_1 = 2n + 1$, the elementary divisors are simple; 2) $2\lambda_1 = 2n + 1$, the elementary divisors are multiple; 3) $2\lambda_1 = 2n$, the elementary divisors are simple; 4) $2\lambda_1 = 2n$, the elementary divisors are multiple. Here n is an integer which can always be taken as zero, as we shall see below (see (2.4)).By analogy with autonomous systems we say that second-order resonance obtains in cases 1) and 2) and first-order resonance obtained in cases 3) and 4). The linear transformation $\|q_1p_1\|^T = \mathbf{N}(t)\|q_1'p_1'\|^T$ with a real symplectic matrix $\mathbf{N}(t)$ differentiable and 2π-periodic in t, taking the Hamiltonian $h_2(q_1, p_1)$ to normal form, can be constructed by analogy with /1,2/.

Theorem 1.1. Hamiltonian $h_2(q_1, p_1)$ is taken into one of the following normal forms:

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$$h_2(q_1', p_1') = {}^1/_2\lambda_1(q_1'^2 + p_1'^2) \ (\lambda_1 = {}^1/_2), \quad \mathbf{N}(t) = \mathbf{X}(t)\mathbf{Q}(t), \quad \mathbf{Q}(t) = \begin{Vmatrix} \cos\lambda_1 t & -\sin\lambda_1 t \\ \sin\lambda_1 t & \cos\lambda_1 t \end{Vmatrix} \ \text{(case 1)} \tag{1.4}$$

$$h_2(q_1', p_1') \equiv 0, \quad \mathbf{N}(t) = \mathbf{X}(t) \ (\mathbf{X}(t+2\pi) = \mathbf{X}(t)) \ \text{(case 3)} \tag{1.5}$$

$$h_2(q_1', p_1') = \frac{1}{2}\delta_1 p_1'^2 \ (\delta_1 = \pm 1), \quad \mathbf{N}(t) = \mathbf{X}(t)\mathbf{P}\mathbf{Q}(t), \quad \mathbf{Q}(t) = \begin{Vmatrix} 1 & -\delta_1 t \\ 0 & 1 \end{Vmatrix} \ \text{(case 4)} \tag{1.6}$$

The constant matrix $\mathbf{P}$ is defined by one of the formulas

$$\mathbf{P} = \begin{Vmatrix} \varkappa_{12} & 0 \\ \delta_1 \dfrac{x_{22}-1}{\sqrt{2\pi|x_{12}|}} & \dfrac{1}{\varkappa_{12}} \end{Vmatrix}, \quad \delta_1 = \operatorname{sign} x_{12}, \text{ if } x_{12} \neq 0; \quad \mathbf{P} = \begin{Vmatrix} \delta_1 \dfrac{x_{11}-1}{\sqrt{2\pi|x_{21}|}} & \dfrac{1}{\varkappa_{21}} \\ -\varkappa_{21} & 0 \end{Vmatrix}, \quad \delta_1 = -\operatorname{sign} x_{21}, \text{ if } x_{21} \neq 0$$

where $\varkappa_{jk} = \sqrt{|x_{jk}|/(2\pi)}$, and x_{jk} $(j, k = 1, 2)$ are the elements of matrix $\mathbf{X}(2\pi)$.

Theorem 1.1 is proved by direct verification of the properties of the matrices $N(t)$.

The normal forms (1.4) — (1.6) coincide with the normal forms for autonomous systems (for which λ_1 has the sense of the frequency of the linear oscillations) in the corresponding resonance cases. Let us show that in Hamiltonian systems case 2) is never realized. Assume the contrary: let $\lambda_1 = {}^1/_2 + n$ and let the elementary divisors of the characteristic matrix $\mathbf{X}(2\pi) - \rho\mathbf{E}$ (where $\rho = \exp(2\pi i\lambda_1) = -1$) be multiple. By Liapunov's reducibility theorem such a system necessarily reduces to a constant-coefficient system

$$dq'/dt = a_{11}q' + a_{12}p', \quad dp'/dt = a_{21}q' + a_{22}p' \tag{1.7}$$

The roots of the defining equation of this system must be definition be pure imaginary. Hence $a_{11} + a_{22} = 0$ and, consequently, (1.7) is a canonic system. But the Hamiltonian of any one-dimensional autonomous canonic system with multiple elementary divisors reduces to form (1.6) wherein the fundamental matrix $\mathbf{Q}(t)$ $(\mathbf{Q}(0) = \mathbf{E})$ has a double eigenvalue $\rho = 1$ when $t = 2\pi$. The fundamental matrix of the original system is similar to $\mathbf{Q}(t)$ since $\mathbf{X}(t) = \mathbf{N}(t)\mathbf{Q}(t)\mathbf{N}^{-1}(t)$. But similar matrices must have like eigenvalues. Consequently, the eigenvalues of matrix $\mathbf{X}(2\pi)$ also equal one, which contradicts the initial assumption $\lambda_1 = {}^1/_2 + n$ $(\rho = -1)$. Therefore, case 2) need not be examined.

Henceforth we reckon that the linear normalization has already taken place and that the quadratic part of Hamiltonian (1.2) has the normal form (1.3) in which $h_2(q_1, p_1)$ is defined by (1.4) — (1.6) for cases 1), 3), 4), respectively. The stability of a one-dimensional system in a nonlinear setting was investigated in /5 — 7/ (also see survey /8/) for various interesting special cases. The most important results were obtained in /6,7/. The case of multidimensional Hamiltonian systems has almost not been considered. The results in the present paper generalize those metioned. In general, it suffices to consider a system with two degrees of freedom and then to carry all results easily over to the case of $n+1$ degrees of freedom if only the characteristic exponents $\pm i\lambda_2, \ldots, \pm i\lambda_{n+1}$ are not connected by parametric resonance relations of combinational or basic type.

2. Let us consider the stability question for system (1.1) in case 1). In the system we make a nonlinear normalization such that the new Hamiltonian K acquires a simpler form. For this we first pass to the complex variables q_k^*, p_k^* by the formulas $(\delta_1 = 1)$

$$q_k^* = \frac{1}{\sqrt{2}}(-\delta_k q_k + i p_k), \quad p_k^* = \frac{1}{\sqrt{2}}(i q_k - \delta_k p_k) \quad (k = 1, 2) \tag{2.1}$$

In the complex variables we have $H_2^* = i\lambda_1 q_1^* p_1^* + i\lambda_2 q_2^* p_2^*$, where $\lambda_1 = {}^1/_2$ in the case being examined, while the coefficients of form H_m^* satisfy the realness relations

$$h^*_{\mu_1\mu_2\nu_1\nu_2} = i^m \delta_1^{\nu_1+\mu_1} \delta_2^{\nu_2+\mu_2} \bar{h}^*_{\nu_1\nu_2\mu_1\mu_2} \tag{2.2}$$

Then the coefficients of the generating function S^* normalizing the polynomial substitution must be the solutions, 2π-periodic in t, of the differential equations /2/

$$(d/dt + i r_{\nu_1\nu_2\mu_1\mu_2}) s^*_{\nu_1\nu_2\mu_1\mu_2} = k^*_{\nu_1\nu_2\mu_1\mu_2} - g^*_{\nu_1\nu_2\mu_1\mu_2}, \quad r_{\nu_1\nu_2\mu_1\mu_2} = \lambda_1(\nu_1 - \mu_1) + \lambda_2(\nu_2 - \mu_2) \tag{2.3}$$

where $g^*_{\nu_1\nu_2\mu_1\mu_2}(t)$ are the coefficients of form G_m^* defined uniquely by recurrence formulas from the coefficients of the terms of lower order (*). From (2.3) we see that if $r_{\nu_1\nu_2\mu_1\mu_2} \neq 0 \pmod 1$, then we can set $k^*_{\nu_1\nu_2\mu_1\mu_2}(t) \equiv 0$. If $r_{\nu_1\nu_2\mu_1\mu_2}$ is an integer, we cannot suppress the corresponding term in the new Hamiltonian, in general. However, we can so choose $s^*_{\nu_1\nu_2\mu_1\mu_2}(t)$ that only the

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*) Markeev, A. P. and Sokol'skii, A. G., Some computational normalization algorithms for Hamiltonian systems. Preprint No.31, Inst. Prikl. Mat. Akad. Nauk SSSR, 1976.

resonant harmonic remains in the Taylor series expansion of $k^*_{\nu_1\nu_2\mu_1\mu_2}(t)$ To be precise, we can set

$$k^*_{\nu_1\nu_2\mu_1\mu_2}(t) = \varkappa_{\nu_1\nu_2\mu_1\mu_2}\exp(-ir_{\nu_1\nu_2\mu_1\mu_2}t), \quad \varkappa_{\nu_1\nu_2\mu_1\mu_2} = a_{\nu_1\nu_2\mu_1\mu_2} + ib_{\nu_1\nu_2\mu_1\mu_2} = \frac{1}{2\pi}\int_0^{2\pi} g^*_{\nu_1\nu_2\mu_1\mu_2}(t)\exp(ir_{\nu_1\nu_2\mu_1\mu_2}t)\,dt \tag{2.4}$$

Here the numbers $\varkappa_{\nu_1\nu_2\mu_1\mu_2}$ possess property (2.2) and is unchanged under the substitution $\lambda_k \to \lambda_k + n$ (n is an arbitrary integer). Thus, after a nonlinear normalization up to terms of order m the Hamiltonian takes the form

$$K^* = i\lambda_1 Q_1{}^* P_1{}^* + i\lambda_2 Q_2{}^* P_2{}^* + \sum \varkappa_{\nu_1\nu_2\mu_1\mu_2}\exp(-ir_{\nu_1\nu_2\mu_1\mu_2}t)\, Q_1^{*^{\nu_1}} Q_2^{*^{\nu_2}} P_1^{*^{\mu_1}} P_2^{*^{\mu_2}} + K^*_{m+1} + \ldots \tag{2.5}$$

where the summation is carried out over nonnegative indices $\nu_1, \nu_2, \mu_1, \mu_2$ such that $3 \leqslant \nu_1 + \nu_2 + \mu_1 + \mu_2 \leqslant m$, while $r_{\nu_1\nu_2\mu_1\mu_2} = n$ (integers). Finally, in (2.5) we pass to real polar variables (φ_k are coordinates, $r_k \geqslant 0$ are momenta) by the formulas

$$Q_k^* = i\sqrt{r_k}\exp[i(\delta_k\varphi_k + \lambda_k t)], \; P_k{}^* = -\delta_k\sqrt{r_k}\exp[-i(\delta_k\varphi_k + \lambda_k t)] \tag{2.6}$$

The stability problems for the original system with respect to variables q_k, p_k and for the normalized system with respect to variables r_k are equivalent.

We restrict the analysis to terms of up to fourth order inclusive ($m = 3, 4$). The normal form will be different in the following four subcases (the n are integers): 1a) $3\lambda_2 \neq n$, $4\lambda_2 \neq 2n+1$, $6\lambda_2 \neq 2n+1$; 1b) $3\lambda_2 = n$; 1c) $4\lambda_2 = 2n+1$; 1d) $6\lambda_2 = 2n+1$.
In subcase 1a) the normal form is:

$$K = K^{(0)} + K^{(1)} \tag{2.7}$$

$$K^{(0)} = \Phi_{40}(\varphi_1) r_1{}^2 + \Phi_{22}(\varphi_1) r_1 r_2 + \Phi_{04} r_2{}^2, \quad K^{(1)} = K_5 + \ldots \tag{2.8}$$

$$\Phi_{40}(\varphi_1) = 2a_{4000}\cos 4\varphi_1 - 2\delta_1 b_{4000}\sin 4\varphi_1 - 2\delta_1 b_{3010}\cos 2\varphi_1 - 2a_{3010}\sin 2\varphi_1 - a_{2020}$$

$$\Phi_{22}(\varphi_1) = -2\delta_2 b_{2101}\cos 2\varphi_1 - 2\delta_1\delta_2 a_{2101}\sin 2\varphi_1 - \delta_1\delta_2 a_{1111}, \quad \Phi_{04} = -a_{0202}$$

Theorem 2.1. 1) If a value $\varphi_1{}^* \in [0, 2\pi]$ exists such that $\Phi_{40}(\varphi_1{}^*) = 0$, while $\Phi_{40}{}'(\varphi_1{}^*) \neq 0$, then the equilibrium position is Liapunov-unstable. 2) If $\Phi_{40}(\varphi_1) \neq 0$ for any real φ_1, then the equilibrium position is stable when terms of up to fourth order, inclusive, are taken in Hamiltonian (1.2). 3) If $\Phi_{40}(\varphi_1) \neq 0$ and the original system has one degree of freedom, then its equilibrium position is Liapunov-stable. 4) If for all φ_1 the function $K^{(0)}$ is sign-definite for $r_1 \geqslant 0$, $r_2 \geqslant 0$, then formal stability obtains.

The instability is proved by constructing the Chetaev function /1,2,4/

$$V = [r_1{}^\alpha - r_2{}^2]\sin\Psi, \quad \Psi = \frac{\pi}{2\varepsilon}(\varphi_1 - \varphi_1{}^* + \varepsilon), \quad 2 < \alpha < 3 \tag{2.9}$$

where, by using the periodicity of $\Phi_{40}(\varphi_1)$, we can so select ε that the inequality $\Phi_{40}{}'(\varphi_1) < 0$ is fulfilled in the neighborhood $|\varphi_1 - \varphi_1{}^*| < \varepsilon$. Then in the region

$$V > 0: \; \{|\varphi_1 - \varphi_1{}^*| < \varepsilon, \; r_2 = \beta r_1{}^{\alpha/2}, \; 0 < \beta < 1\}$$

the derivative of function (2.9) relative to the equations of motion with Hamiltonian (2.7)

$$\frac{dV}{dt} = r_1^{\alpha+1}\left[\frac{\pi}{\varepsilon}(1-\beta^2)\Phi_{40}(\varphi_1)\cos\Psi - \alpha\Phi_{40}(\varphi_1)\sin\Psi\right] + o(r_1^{\alpha+1})$$

is positive definite /4/, whence by Chetaev's theorem we obtain the instability of the equilibrium position.

Since $r_2 = \mathrm{const}$ is an integral of the truncated system with Hamiltonian $K^{(0)}$, we have that $G = sr_2 + K^{(0)}$, where $s = \operatorname{sign}\Phi_{40}(\varphi_1)$ too is an integral of the truncated system, i.e., $dG/dt = 0$, and this integral is sign-definite. Hence by Liapunov's stability theorem (G is the Liapunov function) we obtain the stability of the complete system in the fourth order. If $k\lambda_2 \neq n$, where $k = 3, \ldots, 2m+1$, then from this follows even stability in the m-th order, and, for an irrational λ_2, formal stability /9/).

If the original system is one-dimensional and $\Phi_{40}(\varphi_1) \neq 0$, then by Theorem 2.1 from /4/ (passage to the variables action-angle and use of Moser's theorem on invariant curves) we obtain the Liapunov-stability of the equilibrium position. To prove formal stability we note that after the above-described nonlinear normalization has been carried out for terms up to infinite order, the function (2.7) does not depend explicitly on time, i.e., when the theorem's hypotheses are fulfilled we have a sign-definite formal integral. Then, according to the definition in /9/, the equilibrium position is formally stable, i.e. stable in any finite order. In concluding the proof of Theorem 2.1 we note that its hypotheses are easily verified in a concrete mechanical system. After the substitution $x = \cos 2\varphi_1$ the problem is reduced to ascertaining the conditions for the location on segment $[-1, 1]$ of the roots of a fourth-degree algebraic equation, which can be solved in radicals. However, it is convenient to use an indirect method of the type of Sturm's method.

For subcase 1b)

$$K^{(0)} = \Phi_{03}(\varphi_2)\, r_2^{3/2},\ K^{(1)} = K_4 + \ldots, \qquad \Phi_{03}(\varphi_2) = 2b_{0300}\cos 3\varphi_2 + 2\delta_2 a_{0300}\sin 3\varphi_2 \tag{2.10}$$

in the normal form (2.8).

Theorem 2.2. If in (2.10) $a_{0300}^2 + b_{0300}^2 \neq 0$, then the equilibrium position is unstable.

For subcase 1c) we have

$$\begin{gathered} K^{(0)} = \Phi_{12}(\varphi_1, \varphi_2) r_1^{1/2} r_2,\ K^{(1)} = K_4 + \ldots \\ \Phi_{12} = 2b_{1200}\cos(\varphi_1 + 2\delta_1\delta_2\varphi_2) + 2\delta_1 a_{1200}\sin(\varphi_1 + 2\delta_1\delta_2\varphi_2) + \\ 2\delta_1 a_{0120}\cos(\varphi_1 - 2\delta_1\delta_2\varphi_2) + 2b_{0120}\sin(\varphi_1 - 2\delta_1\delta_2\varphi_2) \end{gathered} \tag{2.11}$$

Theorem 2.3. If in (2.11) $(a_{1200}^2 + b_{1200}^2 - a_{0120}^2 - b_{0120}^2)\,\delta_1\delta_2 > 0$, then the equilibrium position is unstable.

Theorems 2.2 and 2.3 can be proved by using Chetaev's theorem analogously as in /1,2,4/ and Theorem 2.1, having observed that for any values of the coefficients of functions Φ_{03} and Φ_{12} (not vanishing simultaneously) these functions will take values of both signs. We merely remark that case 1c) is equivalent to the simultaneous fulfillment of the resonance relations $\lambda_1 + 2\lambda_2 = n_1$ and $\lambda_1 - 2\lambda_2 = n_2$, where n_1, n_2 are integers of different parity. For subcase 1d) we obtain

$$K^{(0)} = \Phi_{40}(\varphi_1)\, r_1^2 + \Phi_{22}(\varphi_1)\, r_1 r_2 + \Phi_{13}(\varphi_1, \varphi_2)\, r_1^{1/2} r_2^{3/2} + \Phi_{04} r_2^2, \quad K^{(1)} = K_5 + \ldots \tag{2.12}$$

$$\Phi_{13} = 2a_{1300}\cos(\varphi_1 + 3\delta_1\delta_2\varphi_2) - 2\delta_1 b_{1300}\sin(\varphi_1 + 3\delta_1\delta_2\varphi_2) - 2\delta_1 b_{0310}\cos(\varphi_1 - 3\delta_1\delta_2\varphi_2) + 2a_{0310}\sin(\varphi_1 - 3\delta_1\delta_2\varphi_2)$$

Theorem 2.4. 1) If a value $\varphi_1^* \in [0, 2\pi]$ exists such that $\Phi_{40}(\varphi_1^*) = 0$, while $\Phi_{40}'(\varphi_1^*) \neq 0$, then the equilibrium position is unstable. 2) If for $0 \leqslant \varphi_1 < 2\pi,\ 0 \leqslant \varphi_2 < 2\pi,\ r_1 \geqslant 0,\ r_2 \geqslant 0$ the function $K^{(0)}$ is sign-definite, then the equilibrium position is formally stable.

3. Let us consider case 3), when $\lambda_1 = 0$ and the characteristic matrix has simple elementary divisors. We remark that from the applied viewpoint this case is less interesting than the case of multiple elementary divisors, considered in Sect.4, since to realize it the fulfillment of additional conditions is necessary on the elements of matrix $\mathbf{X}(2\pi)$, which leads to $\operatorname{rg}[\mathbf{X}(2\pi) + \mathbf{E}]$ diminishing by one. Therefore, here we limit ourselves to only a brief description of the main results. Under an analysis based on terms of up to fourth order three subcases are possible (the n are integers):

$$\text{3a) } 3\lambda_2 \neq n,\ 4\lambda_2 \neq 2n + 1; \ \text{ 3b) } 3\lambda_2 = n; \quad \text{3c) } 4\lambda_2 = 2n + 1$$

For subcase 3a), in the normal form (2.7)

$$K^{(0)} = \Phi_{30}(\varphi_1)\, r_1^{3/2} + \Phi_{40}(\varphi_1)\, r_1^2 + \Phi_{22}(\varphi_1)\, r_1 r_2 + \Phi_{04} r_2^2$$

$$\Phi_{30}(\varphi_1) = 2b_{3000}\cos 3\varphi_1 - 2\delta_1 a_{3000}\sin 3\varphi_1 + 2\delta_1 a_{2010}\cos\varphi_1 - 2b_{2010}\sin\varphi_1$$

while the remaining functions are defined in (2.8). In formulas (2.4), from which the quantities $a_{\nu_1\nu_2\mu_1\mu_2}$, $b_{\nu_1\nu_2\mu_1\mu_2}$ are computed, we need to set $\lambda_1 = 0$, i.e., in (2.3) $r_{\nu_1\nu_2\mu_1\mu_2} = \lambda_2(\nu_2 - \mu_2)$.

Theorem 3.1. If $a_{3000}^2 + b_{3000}^2 + a_{2010}^2 + b_{2010}^2 \neq 0$, then the equilibrium position is unstable. However, if $\Phi_{30}(\varphi_1) \equiv 0$, then Theorem 2.1 is valid.

The first assertion in Theorem 3.1 can be proved by using the Chetaev function (2.9). Henceforth, we take $\Phi_{30}(\varphi_1) \equiv 0$. Subcase 3b) is completely analogous to subcase 1b), and Theorem 2.2 is valid as well. Subcase 3c) is analogous to subcase 1d). Now the normal form is defined by expressions (2.7), (2.8), wherein

$$\Phi_{04} = \Phi_{04}(\varphi_2) = 2a_{0400}\cos 4\varphi_2 - 2\delta_2 b_{0400}\sin 4\varphi_2 - a_{0202}$$

Theorem 2.4 remains valid. In addition, to it we now can add on the statement: 3) If a value $\varphi_2^* \in [0, 2\pi]$ exists such that $\Phi_{04}(\varphi_2^*) = 0$, while $\Phi_{04}'(\varphi_2^*) \neq 0$, then the equilibrium position is unstable. It can be proved by using the Chetaev function (2.9) in which the subscripts 1 and 2 must be interchanged.

4. Now let $\lambda_1 = 0$, while the elementary divisors are multiple. We note that in contrast to the previously-considered cases, the motion of the linear system is unstable. However, as in the autonomous problem /4/, from such instability (the solution grows as a linearly function of time) there still does not follow the instability of the complete nonlinear system (see /7/ as well).

To carry out the nonlinear normalization we introduce the complex variables q_2^*, p_2^* by formulas (2.1) and we leave the variables q_1, p_1 unchanged, denoting them now by q_1^*, p_1^*. In the complex variables now $H_2^* = {}^1/_2\delta_1 p_1^{*2} + i\lambda_2 q_2^* p_2^*$, while instead of (2.2) we now have the realness conditions

$$h^*_{\nu_1\mu_1\mu_2\nu_2} = (i\delta_2)^{\nu_2+\mu_2}\,\bar{h}^*_{\nu_1\nu_2\mu_1\mu_2} \tag{4.1}$$

Then the equations for determining the coefficients of the generating function and the new Hamiltonian are

$$\left(\frac{d}{dt} + ir_{\nu_1\nu_2\mu_1\mu_2}\right) s^*_{\nu_1\nu_2\mu_1\mu_2} + \delta_1(\nu_1+1)\, s^*_{\nu_1+1,\,\nu_2,\,\mu_1-1,\,\mu_2} = k^*_{\nu_1\nu_2\mu_1\mu_2} - g^*_{\nu_1\nu_2\mu_1\mu_2}, \quad r_{\nu_1\nu_2\mu_1\mu_2} = \lambda_2(\nu_2-\mu_2) \tag{4.2}$$

From (4.2) we see that in K^* we can suppress all terms except those for which $r_{\nu_1\nu_2\mu_1\mu_2} = n$ (integers) and $\mu_1 = 0$ simultaneously. The coefficients of the other terms are determined by formulas (2.4) in which $r_{\nu_1\nu_2\mu_1\mu_2} = \lambda_2(\nu_2-\mu_2)$, while the constants $\varkappa_{\nu_1\nu_2\mu_1\mu_2}$ satisfy the realness conditions (4.1). Then, having further made the substitution (2.6) for the variables with subscript 2 and omitting the asterisk on the variables with subscript 1, we obtain a real normal form of the Hamiltonian. Let $k\lambda_2 \neq n$ (the n are integers) for $k = 3, \ldots, m$. In this case, similarly to the autonomous problem /4/, we have

$$K = \frac{1}{2}\delta_1 P_1^2 + \sum_{k=3}^{m}\sum_{l=0}^{[k/2]} A_{k-2l,\,2l}\, Q_1^{k-2l} r_2^{\,l} + K_{m+1} + \ldots$$

$$A_{k-2l,\,2l} = \begin{cases} (-1)^L a_{k-2l,\,l,\,0,\,l}, & l = 2L, \quad L = 0, 1, 2, \ldots \\ (-1)^L \delta_2 b_{k-2l,\,l,\,0,\,l}, & l = 2L+1 \end{cases}$$

where it is assumed that normalization has been carried out up to an order m such that $A_{m,0} \neq 0$.

Theorem 4.1. 1) If m is odd, then the equilibrium position is unstable. 2) If m is even and $\delta_1 A_{m,0} < 0$, then the equilibrium position is unstable. 3) If m is even and $\delta_1 A_{m,0} > 0$, then the equilibrium position is stable when terms of up to order m are taken into account. 4) If m is even, $\delta_1 A_{m,0} > 0$ and $\delta_1 A_{0,2} > 0$, then the equilibrium position is formally stable. 5) If m is even, $\delta_1 A_{m,0} > 0$ and the system has one degree of freedom, then its equilibrium position is Liapunov-stable.

The proof of this theorem is obtained by combining the proofs of Theorem 4.1 of /4/ and of Theorem 2.1 of the present paper. The subcases $3\lambda_2 = n$, $4\lambda_2 = 2n+1$ and others are investigated analogously as in the preceding sections.

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REFERENCES

1. MARKEEV A. P., Libration Points in Celestial Mechanics and Astrodynamics. Moscow, "Nauka", 1978.
2. IVANOV A.P. and SOKOL'SKII A.G., On the stability of a nonautonomous Hamiltonian system under second-order resonance. PMM Vol.44, No.5, 1980.
3. IAKUBOVICH V.A. and STARZHINSKII V.M., Linear Differential Equations with Periodic Coefficients and Their Applications. Moscow, "Nauka", 1972.
4. SOKOL'SKII A.G., On the stability of an autonomous Hamiltonian system with two degrees of freedom under first-order resonance. PMM Vol.41, No.1, 1977.
5. LEVI-CIVITA T., Sorpa alcuni criteri di instabilita. Ann. mat. pure ed appl., ser.3, Vol. 5, p.221, 1901.
6. MERMAN G.A., On the instability of the periodic solution of a canonic system with one degree of freedom in the case of essential resonance. In: Problems of the Motion of Artificial Celestial Bodies. Moscow, Izd. Akad. Nauk SSSR, 1963.
7. MERMAN G.A., Asymptotic solutions of a canonic system with one degree of freedom in the case of zero characteristic exponents. Biul. Inst. Teoret. Astron. Akad. Nauk SSSR, Vol. 9, No.6 (109), 1964.
8. KUNITSYN A.L. and MARKEEV A. P., Stability in resonance cases. In: Reviews in Science and Engineering. General Mechanics Series. Vol.4, Moscow, VINITI, 1979.
9. MOSER J., New aspects in the theory of stability of Hamiltonian systems. Communs Pure and Appl. Math., Vol.11, No.1, p.81-114, 1958.

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